UNIT 6: ALGEBRAIC EXPRESSIONS:

Algebraic Language:

Algebra is a branch of Mathematics in which symbols, usually letters of the alphabet, represent numbers or members of a specified set and are used to represent quantities and to express general relationships that hold for all members of the set.
What do you do when you want to refer to a number that you do not know? Suppose you want to refer to the number of buildings in your town, but haven't counted them yet. You could say a "blank" number of buildings. But in Mathematics, we often use letters to represent numbers that we do not know – so you could say "x" number of buildings, or "q" number of buildings.

Look at these examples:

The triple of a number:	3n
The triple of a number minus five units:	3n-5
The following number:	x+1
The preceding number:	x-1
An even number:	2x
An odd number:	2x+1

1. Find the expressions:

a) I start with x, double it and the subtract 10.

b) I start with x, add 3 and then square the result.

c) I start with x, take away 5, double the result and then divide by 5.

d) I start with x, multiply by 5 and then subtract 3.

e) I start with x, add y and then double the result.

f) I start with n, square it and then subtract n.

g) I start with x, add 2 and then square the result.

h) A brick weighs x kg. How much do 6 bricks weigh? How much do n bricks weigh?

Algebraic Expressions:

An algebraic expression is a set of numbers and letters join with the signs of arithmetic operations.

Examples:

The perimeter of a rectangle: P=2a+2b
The volume of a cube with edge a: V= a^3
The sum of the squares of two consecutive numbers: $x^2+(x+1)^2$
The 27% of a number C: $\frac{27}{100}\cdot C$

Numerical value of an algebraic expression:

The numerical value of an algebraic expression is the result when the letters are replaced with some determined numbers.

Examples:

a) Calculate the numerical value of the algebraic expression x^2+1 if $x=3$:

$$3^2+1=9+1=10$$

b) Work out the value of the algebraic expression $2a-3b$ when $a=5$ and $b=-1$:

$$2\cdot5-3\cdot(-1)=10+3=13$$

1. Calculate the numerical value of the algebraic expression $2x^2-1$ when:

   $x=3$

   $x=1$

   $x=-2$

   $x=-5$

2. Calculate the numerical value of the expression $2x-y^2+3z+1$ when:

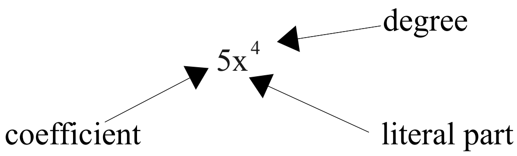
$x=1,\ y=0,\ z=-1$

$x=2,\ y=1,\ z=0$

$x=3,\ y=-2,\ z=1$

Monomials:

A **monomial** is an algebraic expression consisting of only one term, which has a known value (coefficient), and one or some unknown values represented by letters (literal part). For example:



Remember: like terms use exactly the same literal part:

$3x$ and $6x$ are like terms
$3x$ and $3xy$ are unlike terms
$3x$ and $3x^2$ are unlike terms.

Addition and subtraction of monomials:

You can add monomials only if they have the same literal part (they are also called **like terms**). In this case, you sum the coefficients and leave the same literal part.

Look at these examples:

$5xy^2+2xy^2=7xy^2$
$5x^2+3-2x^2+1=3x^2+4$

1. Collect like terms to simplify each expressions:

a) $x^2+3x-x+3x^2$

b) $3x-4-(x+1)$

c) $5y+3x+2y+4x$

d) $(2x+3)-(5x-7)-(x+1)$

e) $\frac{2}{3}x^2+\frac{1}{2}x-\frac{3}{2}x^2-\frac{1}{5}x+2$

Product of monomials:

If you want to multiply two or more monomials, you just have to multiply the coefficients, and add the exponents of the equal letters:

Look at these examples:

$5x^2 \cdot 3x^4 = 15x^6$

$(2xy^2) \cdot (-5x^2 y) = -10x^3 \cdot y^3$

$3a^3 \cdot 2ab = 6a^3 b$

Quotient of monomials:

If you want to divide two monomials, you just have to divide the coefficients, and subtract the exponents of equal letters. You can also simplify the fractions which result from the division. Look at these examples:

$(10x^3):(2x) = 5x^2$

$(8x^2 y):(6y^3) = \frac{8x^3 y}{6y^3} = \frac{2 \cdot 4 \cdot x \cdot x \cdot y}{2 \cdot 3 \cdot y \cdot y \cdot y} = \frac{4x^2}{3y^2}$

1. Calculate:

a) $-2x^2+x+3x^2+x^3+x$
b) $3x^3-(2x^3-4x^3)$
c) $8x^3-x^2+9x^2+x^3$
d) $7xy^2-4x^2y-x^2y-2xy^2$
e) $x^2 \cdot x^3 \cdot x^6$
f) $3x^2 \cdot 2x^4 \cdot 5x$

g) $-2x^2 \cdot 5x^2 \cdot x^2$

h) $\frac{2}{5}x^3 \cdot 4x \cdot (-3x^2)$

i) $(12x^7):(4x^2)$

j) $(-10x^5 \cdot y^3):(5x^4 \cdot y)$

k) $\frac{18x^6}{6x^2}$

l) $\frac{20x^4 y^3}{5xy}$

Polynomials:

A polynomial is the addition or subtraction of two or more monomials.

If there are two monomials, it is called a **binomial**, for example: x^2+x .
If there are three monomials, it is called a **trinomial,** for example: $2x^2-3x+1$.

The following algebraic expressions are NOT polynomials:

$\frac{1}{x}$ $\sqrt{x^2-4}$ $x^2+3x+2x^{-2}$

The degree of the entire polynomial is the degree of the highest-degree term that it contains, so

x^2+2x-7 is a second-degree trinomial, and x^4-7x^3 is a fourth-degree polynomial.

The polynomial that follows is a second degree polynomial, and there are three terms: $4x^2$ is the leading term, and -7 is the constant term.

Terms

$4x^2+3x-7$

leading term constant term

Polynomials are usually written this way, with the terms written in "**decreasing**" order; that is, with the highest exponent first, the next highest next, and so forth, until you get down to the constant term.
Polynomials are also sometimes named for their degree:
a second-degree polynomial, such as $4x^2$, x^2-9 , or ax^2+bx+c , is also called a "**quadratic**".
A third-degree polynomial, such as $-6x^3$ or x^3-27 , is also called "**cubic**".
A fourth-degree polynomial, such as x^4 or $2x^4-3x^2+9$, is sometimes called a "**quartic**".

Evaluating Polynomials:

"Evaluating" a polynomial is the same as calculating its number value at a given value of the variable: you plug in the given value of *x*, and figure out what the polynomial is supposed to be. For instance, let's evaluate $2x^3 - x^2 - 4x + 2$ at $x = -3$.

Substitute *x* for -3, remembering to be careful with the brackets and negatives:

$$2(-3)^3 - (-3)^2 - 4(-3) + 2 = 2(-27) - 9 + 12 + 2 = -54 - 9 + 12 + 2 = -49$$

1. Evaluate the polynomial $x^3 - 2x^2 + 3x - 4$ at the given value of *x*:

$x = 0$

$x = 1$

$x = 2$

$x = -1$

2. Evaluate the polynomial $x^4 - 2x^3 + x^2 - 5$ at the following values of *x*:

$x = 2$

$x = 0$

$x = -1$

$x = -2$

Adding polynomials:

When adding polynomials, you use what you know about the addition of monomials. But there are two ways of doing it. The format you use, horizontal or vertical, is a matter of taste (unless the instructions explicitly tell you otherwise). Given a choice, you should use the format that you prefer. Note that for simple addition (so you don't have to rewrite the problem) is probably the simplest, but, once the polynomials get complicated, vertical addition is probably the safest (so you don't "drop", or lose, terms and minus signs). Here is an example:

Simplify $(3x^3+3x^2-4x+5)+(x^3-2x^2+x-4)$

Vertically:

$$\begin{array}{rrrrr} & 3x^3 & +3x^2 & -4x & +5 \\ + & x^3 & -2x^2 & +x & -4 \\ \hline & 4x^3 & +x^2 & -3x & +1 \end{array}$$

Horizontally:

$(3x^3+3x^2-4x+5)+(x^3-2x^2+x-4)=3x^3+3x^2-4x+5+x^3-2x^2+x-4=4x^3+x^2-3x+1$

Either way, you get the same answer: $4x^3+x^2-3x+1$.

1. Given the polynomials: $P(x)=x^4-3x^3+6x^2+3x-4$; $Q(x)=x^3-4x^2+x-2$; $R(x)=2x^3+x^2-5x+4$; calculate:

a) $P(x)+Q(x)$ b) $P(x)+R(x)$ c) $Q(x)+R(x)$

Subtracting Polynomials:

When subtracting polynomials, you must realize that a subtraction is the addition of the first term and the opposite of the second: A-B=A+(-B).

Notice that running the negative trough the brackets changes the sign on each term inside the brackets. Look at this example:

Simplify: $(6x^3-2x^2+8x)-(4x^3-11x+10)$:

Vertically:

$$\begin{array}{rrrr} 6x^3 & -2x^2 & +8x & \\ -4x^3 & & +11x & -10 \\ \hline 2x^3 & -2x^2 & +19x & -10 \end{array}$$

Horizontally:

$$(6x^3-2x^2+8x)-(4x^3-11x+10)=6x^3-2x^2+8x-4x^3+11x-10=2x^3-2x^2+19x-10$$

Either way, you get the same answer: $2x^3-2x^2+19x-10$.

1. Given the polynomials $P(x)=2x^4-3x^3-x^2+4x-6$; $Q(x)=x^4-2x^3+7x-2$; $R(x)=x^3-x^2+4$; calculate:

a) $P(x)-Q(x)$ b) $P(x)-R(x)$ c) $Q(x)-R(x)$

Multiplying Polynomials:

Monomial times a multi-term polynomial. I distribute the monomial trough the brackets. For example:

$$-3x(4x^2-x+10)=-3x(4x^2)-3x(-x)-3x(10)=-12x^3+3x^2-30x$$

Multi-term polynomial times a multi-term polynomial. Look at these examples:

Simplify: $(x+3)\cdot(x+2)$

The first way I can do this is "horizontally", where I apply the distributive property twice:

$$(x+3)\cdot(x+2)=x\cdot x+x\cdot 2+3\cdot x+3\cdot 2=x^2+2x+3x+6=x^2+5x+6$$

The "vertical" method is much simpler, because it is similar to the multiplications learnt at primary school:

Simplify: $(x-4)(x-3)$

$$\begin{array}{r} x-4 \\ x-3 \\ \hline -3x+12 \\ x^2-4x \\ \hline x^2-7x+12 \end{array}$$

Simplify:

$$\begin{array}{r} 4x^2-4x-7 \\ x+3 \\ \hline 12x^2-12x-21 \\ 4x^3-4x^2-7x \\ \hline 4x^3+8x^2-19x-21 \end{array}$$

1. Given the polynomials: $P(x)=x^2+5x-3$ and $Q(x)=x-2$, calculate their product: $P(x)\cdot Q(x)$.

2. Calculate the following additions of polynomials:

a) $(14x+5)+(10x+5)$

b) $(10x+12)+(6x+20)$

c) $(19x^2+12x+12)+(7x^2+10+13)$

d) $(13x^2-5x-1)+(-14x^2-20x+8)$

3. Calculate the following subtractions:

a) $(6x+14)-(9x+5)$

b) $(6x+19)-(14x+5)$

c) $(14x^2+13x+12)-(7x^2+20x+4)$

d) $(-9x^2-4x-4)-(-9x^2-11x+12)$

4. Remove brackets:

a) $(8x+11)(5x+11)$

b) $(11x+5)(-11x+12)$

c) $(8x+11)(-3x+6)$

d) $(4x^2+12x+10)(-9x^2+8x+2)$

e) $(-12x-3)(12x^2-11x+3)$

Division of a polynomial by a monomial:

We divide each term of the polynomial by the monomial:

Look at the examples:

$$(12x^6-3x^4-9x^2+6x):3x=4x^5-x^3-9x+6$$

$$(6x^4-4x^3+2x^2):(2x^2)=3x^2-2x+1$$

1. Calculate:

a) $(-10x^6+4x^5+8x^3-2x^2):(-2x^2)$

b) $(-25x^5+45x^4+60x^3):(5x)$

Factorising:

Factorising is the reverse process of multiplying out a bracket. The factorised expression has a polynomial inside a bracket, and a term outside.
This term outside must be a common term (a number or a letter). It means that the number of the letter can be found in every term of the expression.
The trick is to see what can be factored out of every term in the expression. Just don't make the mistake of thinking that "factoring" means "dividing off and making disappear". Nothing disappears when you factor; things just get rearrenged. Here are same examples of how to factor:

$$3x-12=3(x-4)$$

$$12y^2-5y=y(12y-5)$$

$$3x^3+6x^2-15x=3x(x^2+2x-5)$$

Remember: when the term to be factored out coincides with one of the addens, the unit always remains:

$$x^2+x=x(x+1)$$

Factorising helps us when we want to simplify algebraic fractions, for exmple:

$$\frac{3x+3y}{x^2+x^3}=\frac{3\cdot(x+y)}{x\cdot(x+y)}=\frac{3}{x}$$

$$\frac{x^2}{x^2+x^3}=\frac{x^2\cdot 1}{x^2\cdot(1+x)}=\frac{1}{1+x}$$

1. Factorise:

a) $20x^3+12x$

b) $3x^2-3x$

c) $3x^3+33x$

d) $16x^3+10x^2-18x$

e) $19x^3-19x$

2. Reduce the following algebraic fractions to their lowest terms, factorising first:

a) $\frac{5a+5b}{5a+10}$

b) $\frac{6x^3}{4x^2+2x^2}$

c) $\frac{x+x^2}{x^2+x^3}$

d) $\frac{6x^3}{2x^3+4x^2}$

Special Products:

Some products occur so frequently in Algebra that is advantageous to be able to recognize them by sight. This will be particularly useful when we talk about factoring. This year we are going to learn three kinds of special products:

a. **THE SQUARE OF A SUM:** $(a+b)^2 = a^2 + 2ab + b^2$

Expand as in the examples:

- $(x+2)^2 = x^2 + 2 \cdot x \cdot 2 + 2^2 = x^2 + 4x + 4$
- $(2x+3)^2 = (2x)^2 + 2 \cdot 2x \cdot 3 + 3^2 = 4x^2 + 12x + 9$
- $(x+1)^2$
- $(x+y)^2$
- $(x+3)^2$
- $(x+4)^2$
- $(2x+1)^2$

b. **THE SQUARE OF A DIFFERENCE:** $(a-b)^2 = a^2 - 2ab + b^2$

Expand as in the examples:

- $(x-5)^2 = x^2 - 2 \cdot x \cdot 5 + 5^2 = x^2 - 10x + 25$
- $(1-2x)^2 = 1^2 - 2 \cdot 1 \cdot 2x + (2x)^2 = 1 - 4x + 4x^2$
- $(x-1)^2$
- $(x-4)^2$
- $(3x-5)^2$
- $(2a-1)^2$
- $(x-3)^2$

c. **PRODUCT OF A SUM AND A DIFFERENCE:** $(a+b)\cdot(a-b)=a^2-b^2$

Expand as in the examples:

- $(x+2)\cdot(x-2)=x^2-2^2=x^2-4$
- $(2x+3)\cdot(2x-3)=(2x)^2-3^2=4x^2-9$
- $(a+1)\cdot(a-1)$
- $(x+4)\cdot(x-4)$
- $(2x+1)\cdot(2x-1)$
- $(2a+3b)\cdot(2a-3b)$
- $(x+3)\cdot(x-3)$

You should be able to recognise these products both ways. That is, if you see the left side you should think of the right side, and if you see the right side you should think of the left side. This is useful when we want to factorise some algebraic expressions or when we want to express some algebraic fractions in their lowest term. For example:

$$x^2-6x+9=x^2-2\cdot x\cdot 3+3^2=(x+3)^2$$

$$x^2-9=x^2-3^2=(x+3)\cdot(x-3)$$

$$x^2+12x+36=x^2+2\cdot 6\cdot x+6^2=(x+6)^2$$

$$\frac{x^2-6x+9}{x^2-9}=\frac{(x+3)^2}{(x+3)\cdot(x-3)}=\frac{x-3}{x+3}$$

1. Factor completely:

x^2-25	$x^2-18x+81$
x^2-9	$x^2-20x+100$
x^2-81	$x^2-14x+49$
x^2-100	$x^2+22x+121$

x^2-49	$x^2+18x+81$
$9x^2-16$	$16x^2-9$
x^2-36	$25x^2-120x+144$
$49x^2+84x+36$	$x^2-16x+64$
$81x^2-180x+100$	$49x^2-64$
x^2-1	$25x^2-1$
$4x^2-12x+9$	x^2-6x+9
$x^2+10x+25$	x^2-121

2. Reduce the following algebraic fractions to their simplest forms:

a) $\dfrac{6x-4}{9x^3-6x^2}$

b) $\dfrac{5x^2+10x}{x+2}$

c) $\dfrac{x^3+x^2}{2x^3-3x^2}$

d) $\dfrac{3x^3-x^2}{x^3+2x^2}$

e) $\dfrac{x^2+2x+1}{x^2-1}$

f) $\dfrac{x^2-4}{x^2-4x+4}$

g) $\dfrac{2x^2-8}{x+2}$

h) $\dfrac{2x+1}{4x^2+4x+1}$

i) $\dfrac{2x^4-2x^3}{4x^4-4x^2}$

j) $\dfrac{3x^4-9x^2}{x^2-3}$

Keywords:

Algebra= Álgebra
Algebraic language= lenguaje algebraico
Algebraic expression= expresión algebraica
monomial= monomio
coefficient= coeficiente
literal part= parte literal
degree= grado
like terms/like monomials= términos semejantes/monomios semejantes
Polynomial= Polinomio
binomial= binomio
trinomial= trinomio
terms= términos
constant term= término independiente
quadratic (second-degree polynomial)= polinomio de segundo grado
cubic (third-degree polynomial)= polinomio de tercer grado
quartic (fourth-degree polynomial)= mpolinomio de cuarto grado
Numerical value= valor numérico
to factorize= factorizar (sacar factor común)
Algebraic fraction= Fracción algebraica